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Photon energy spectrum of electron–positron bremsstrahlung in the center-of-mass system

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Abstract. Formulae for the total cross section of electron–positron bremsstrahlung in the center-of-mass system are given which are exact in lowest-order perturbation theory. The resulting photon spectra are compared with those of electron–electron bremsstrahlung.

1 Introduction

Bremsstrahlung emission in the collision of electrons and positrons is a fundamental radiation process of quantum electrodynamics. In lowest-order perturbation theory the process is described by eight Feynman diagrams, four of them representing scattering graphs and four representing virtual annihilation graphs [1]. Stimulated by experiments with colliding electron–positron beams, most calculations of the angular distribution and spectrum of electron–positron bremsstrahlung (EPB) were performed in the limit of ultrarelativistic energies [1–7], where only two of the eight Feynman diagrams contribute appreciably. In the non-relativistic and mildly relativistic energy regime the EPB process excited interest in the study of hot astrophysical plasmas [8]. The EPB cross section differential in photon energy and angles, exact in lowest-order perturbation theory, was derived by Haug [9]. Since the formula is expressed in terms of invariant products, it can be specialized to any frame of reference. The integration over the photon angles can no more be performed generally. However, in the center-of-mass system, which is realized in experiments with colliding electron–positron beams, it is possible to carry out analytically nearly all the integrations over the photon angles. A further simplification is reached near the low-energy end of the bremsstrahlung spectrum by expanding the cross section formula into powers of the photon energy yielding an accurate expression for the EPB cross section without any integrals. This may be used to evaluate the bremsstrahlung energy loss of the colliding particles.

In the following, relativistic units are used, i.e., all the energies will be expressed in units of the electron rest energy mc^2 and the momenta in units of mc, unless specified otherwise. Then the relation between the particle energies and momenta is $\epsilon^2 = p^2 + 1$.

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2 Cross sections

Specializing the differential cross section of electron–positron bremsstrahlung (EPB) to the center-of-mass system, most of the integrations over the photon angles can be performed analytically. The result can be written as[1]

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}k} = \frac{\alpha r_0^2}{\epsilon p k}\Bigg\{ & a_1\sqrt{\frac{p^2-\epsilon k}{\epsilon^2-\epsilon k}} \\ & + a_2L_2 + a_3L_3 - a_4L_4 + a_5L_5 + a_6L_2L_3 \\ & + \int_{-1}^{+1}\Bigg[\frac{L_1}{W_1}\Bigg\{b_1 - b_2/\kappa_1 + \frac{1}{W_1^2}\left(b_3/\kappa_1 + b_4/\kappa_2\right) \\ & \quad + \frac{1}{W_1^4}\left(b_5/\kappa_1 + b_6/\kappa_2\right)\Bigg\} \\ & \quad + \frac{L_6}{W_6}\left(b_7 - b_8/\kappa_1 + b_9\kappa_1 - b_{10}\kappa_1^2\right)\Bigg]\mathrm{d}\left(\cos\theta\right)\Bigg\},\end{aligned} \tag{1}$$

where $\alpha \approx 1/137$ denotes the fine structure constant, r_0 is the classical electron radius, k the photon energy, and

$$\begin{aligned}a_1 = & \frac{1}{p^2}\left(9\epsilon k - 2k^2 - 7\right) - 4k^2 - 21 \\ & - \frac{\epsilon - k}{3\epsilon^3}\left(16\epsilon^4 + 19\epsilon^2 - \epsilon k + 8 + 1/2\epsilon^2\right) - \frac{k^2}{6\epsilon^4} \\ & - \frac{k^2+8}{3\left(\epsilon^2-\epsilon k\right)} - \frac{k^2+1}{6\left(\epsilon^2-\epsilon k\right)^2} \\ & - \frac{2\epsilon k + 7}{p^2 - \epsilon k} - \frac{3\epsilon^2 k^2}{4\left(\epsilon - k\right)\left(\epsilon p^2 + k\right)}\end{aligned}$$

[1] Some further integrations such as $\int L_1/(\kappa_2 W_1^5)\,\mathrm{d}(\cos\theta)$ may be done analytically. However, the result is rather lengthy so that it is more convenient to perform the integrations numerically together with the remaining terms

$$+\frac{\ln(\epsilon+p)}{\epsilon p}$$

$$\times\left[17\epsilon^2-8\epsilon k+5k^2-2\epsilon^3 k+6+\frac{1}{p^2}\left(7-9\epsilon k+2k^2\right)\right.$$

$$+\frac{1}{\epsilon^2-\epsilon k}\left(\tfrac{15}{4}\epsilon^2k^2+\tfrac{7}{2}\epsilon^2+k^2-2\right)$$

$$+\left(\epsilon^2-\tfrac{1}{2}\right)\frac{\left(p^2-\epsilon k\right)^2}{3\left(\epsilon^2-\epsilon k\right)^2}-\frac{k^2\left(p^2-\epsilon k\right)}{3\left(\epsilon-k\right)^2}$$

$$-\frac{\epsilon k}{p^2-\epsilon k}\left(2\epsilon^2k^2-\tfrac{1}{4}\epsilon k-k^2+\tfrac{3}{2}\right)+\frac{2\epsilon k p^2}{\left(p^2-\epsilon k\right)^2}$$

$$\left.+\frac{\epsilon k}{2\left(\epsilon p^2+k\right)}\left(3k-\frac{\epsilon^2+p^2}{\epsilon-k}\right)+\frac{3\epsilon^2p^2k^2}{4\left(\epsilon p^2+k\right)^2}\right],$$

$$a_2=2\epsilon k+1+\frac{1}{\epsilon^2}\left(2k^2-9\epsilon k+1-\frac{4k}{\epsilon}-\frac{3}{2\epsilon^2}\right)$$

$$+\frac{p^2}{3\epsilon^6}\left(\epsilon^4-\epsilon^2k^2-\epsilon k p^2+\tfrac{1}{2}\right)+\frac{7}{p^2-\epsilon k}+\frac{1}{\epsilon^2-\epsilon k}$$

$$+\frac{2\epsilon k}{4p^2\left(\epsilon^2-\epsilon k\right)-k^2}\left(4\epsilon^2k^2+k^2+2+\frac{2\epsilon}{\epsilon-k}\right)$$

$$+\frac{\ln(\epsilon+p)}{\epsilon p}\left(16\epsilon k-32\epsilon^2-6k^2+\frac{1}{\epsilon^2}+\frac{1-6\epsilon^2k^2}{\epsilon^2-\epsilon k}\right),$$

$$a_3=\frac{1}{p}\left\{\tfrac{32}{3}\epsilon\left(p^2-\epsilon k\right)+8\epsilon k^2+8\epsilon+\tfrac{10}{3}k+\frac{\epsilon k^2-7k/2}{p^2}\right.$$

$$+\frac{k^2-4/3}{\epsilon}+\frac{5k}{2\epsilon^2}+\frac{p^2k^2}{2\epsilon^3}+\frac{k}{\epsilon^4}+\frac{7k}{2\left(p^2-\epsilon k\right)}$$

$$+\frac{k^3-3k}{2\left(\epsilon^2-\epsilon k\right)}-\frac{k}{2\left(\epsilon^2-\epsilon k\right)^2}$$

$$+\frac{k^2\left(\epsilon-k/2\right)}{\left(\epsilon^2-\epsilon k\right)\left[4p^2\left(\epsilon^2-\epsilon k\right)-k^2\right]}$$

$$\times\left(4\epsilon^2k^2+k^2+2+\frac{2\epsilon}{\epsilon-k}\right)$$

$$+\frac{\ln(\epsilon+p)}{\epsilon p}\left[18\epsilon^2k-8\epsilon p^2-6\epsilon k^2\right.$$

$$-4k-\frac{3k}{2\epsilon^2}-\frac{1}{2\epsilon^3}+k\,\frac{7/2-k/\epsilon}{p^2}+\frac{2}{\epsilon^2 k}\left(\epsilon^2+p^2\right)^3$$

$$\left.\left.+\frac{1}{\epsilon-k}\left(6\epsilon^2k^2-3k^2+4p^2-p^2/\epsilon^2\right)\right]\right\},$$

$$a_4=\frac{k\sqrt{p^2-\epsilon k}}{4p\left(\epsilon^2-\epsilon k\right)}\left\{2-\epsilon^2+\frac{5\epsilon k/2-3-2k^2\left(\epsilon^2+p^2\right)}{p^2-\epsilon k}\right.$$

$$+\frac{4p^2}{\left(p^2-\epsilon k\right)^2}$$

$$+\frac{\epsilon^2}{\epsilon p^2+k}$$

$$\left.\times\left[2\left(\epsilon-k\right)+\frac{\epsilon k+k^2/4-1}{\epsilon-k}-\frac{3\epsilon k p^2}{2\left(\epsilon p^2+k\right)}\right]\right\},$$

$$a_5=\frac{k}{p\sqrt{\left(\epsilon^2-\epsilon k\right)\left(p^2-\epsilon k\right)}}\left\{p^2\left(p^2-\epsilon k-2\right)\right.$$

$$+\epsilon^2k^2+\frac{k}{2\epsilon}+\frac{11\epsilon k+2}{8\left(\epsilon^2-\epsilon k\right)}-\frac{p^2}{p^2-\epsilon k}$$

$$+\frac{1}{4\left(\epsilon p^2+k\right)}$$

$$\left.\times\left[3\epsilon k\left(\epsilon-k\right)+\frac{p^2+k^2}{\epsilon-k}+\frac{3\epsilon k p^2\left(p^2-\epsilon k\right)}{2\left(\epsilon p^2+k\right)}\right]\right\},$$

$$a_6=\frac{1}{p\sqrt{\left(\epsilon^2-\epsilon k\right)\left(p^2-\epsilon k\right)}}\left\{44\epsilon p^2-52\epsilon^2k+28\epsilon k^2\right.$$

$$-6k^3+3\epsilon+11k+\frac{k^2+6}{\epsilon}+\frac{5k}{\epsilon^2}+\frac{1}{\epsilon^3}$$

$$\left.-\frac{2}{\epsilon^2k}\left(\epsilon^2+p^2\right)^3-\frac{7k}{2\left(p^2-\epsilon k\right)}+\frac{k-1/\epsilon}{2\left(\epsilon^2-\epsilon k\right)}\right\},$$

$$b_1=k^2\left\{3-\frac{k}{\epsilon}+\frac{1}{\epsilon^2}+\frac{\epsilon^2+1}{\epsilon^2-\epsilon k}\right\},$$

$$b_2=k\left\{4\epsilon^3+6\epsilon^2k-2\epsilon k^2+k-k^2/\epsilon\right.$$

$$\left.+\frac{1}{\epsilon-k}\left(4\epsilon^2k^2+\epsilon^2-2k/\epsilon+1/2\epsilon^2\right)\right\},$$

$$b_3=4\epsilon k\left\{\epsilon^4\left[12-2k^2-4\left(\epsilon^2-\epsilon k\right)\right]-16\epsilon^3k\right.$$

$$\left.+6\epsilon^2k^2-2\epsilon k^3+5\epsilon^2-2\epsilon k+2k^2-1\right\}$$

$$+k^2\left\{2-4k^2+\frac{\epsilon}{\epsilon-k}\left(2-5\epsilon k\right)\right\},$$

$$b_4=\epsilon k\left\{4\left(\epsilon^2-\epsilon k\right)\left(6\epsilon k p^2-4\epsilon^4-4\epsilon^2k^2+2\epsilon k^3\right.\right.$$

$$\left.\left.+12\epsilon^2+k^2+3\right)+8\epsilon^2+3k^2-4\right\},$$

$$b_5=12\epsilon^2k\left\{4\epsilon\left(3-\epsilon^4\right)\left(\epsilon-k\right)^2\right.$$

$$\left.-\epsilon^2\left(\epsilon-k\right)\left(8\epsilon^2+8\epsilon k-3k^2\right)+\epsilon k^2-2k^3\right\},$$

$$b_6=12\epsilon k\left(\epsilon^2-\epsilon k\right)\left\{4\epsilon k+k^2+\left(12-k^2\right)\left(\epsilon^2-\epsilon k\right)\right.$$

$$\left.-4\left(\epsilon^2+2\right)\left(\epsilon^2-\epsilon k\right)^2\right\},$$

$$b_7=k^2\left(9\epsilon-7k+2k^2/\epsilon-k/2\epsilon^2-1/\epsilon^3\right)$$

$$-k\left(\epsilon^2+p^2\right)\left(2-1/4\epsilon^4\right),$$

$$b_8=2\left(k^2/\epsilon\right)\left(\epsilon^2+p^2\right),$$

$$b_9 = k\left(1 - \frac{k}{\epsilon} - \frac{2\epsilon^2 + \epsilon k + 1}{4\epsilon^4}\right) ,$$

$$b_{10} = \frac{\epsilon - k}{4\epsilon^3}\, k ,$$

$$\kappa_1 = k\,(\epsilon - p\cos\theta) \ , \ \kappa_2 = k\,(\epsilon + p\cos\theta) \ ,$$

$$L_1 = \ln \frac{\left(2p^2 - \kappa_1\right)\sqrt{\epsilon^2 - \epsilon k} + W_1\sqrt{p^2 - \epsilon k}}{\kappa_2} ,$$

$$L_2 = \ln\left(\sqrt{\epsilon^2 - \epsilon k} + \sqrt{p^2 - \epsilon k}\right) ,$$

$$L_3 = \ln\left\{\left(\epsilon\sqrt{p^2 - \epsilon k} + p\sqrt{\epsilon^2 - \epsilon k}\right)^2 \Big/ (\epsilon k)\right\} ,$$

$$L_4 = \ln \frac{2p\sqrt{\epsilon^2 - \epsilon k} + k}{2p\sqrt{\epsilon^2 - \epsilon k} - k} ,$$

$$L_5 = \ln \frac{2\epsilon p + (\epsilon - p)k}{2\epsilon p - (\epsilon + p)k} ,$$

$$L_6 = \ln\Bigg\{1 + \frac{2}{k\kappa_2}\ \Big[(\epsilon - k)\left(p^2 - \epsilon k\right)\kappa_1 + \sqrt{\left(\epsilon^2 - \epsilon k\right)\left(p^2 - \epsilon k\right)}\, W_6\Big]\Bigg\} ,$$

$$W_1 = \sqrt{\kappa_2^2 + 4\left(\epsilon^2 - \epsilon k\right)\left(\kappa_2 + p^2 - \epsilon k\right)} ,$$

$$W_6 = \sqrt{\kappa_1\{\left(p^2 + k^2 - 2\epsilon k\right)\kappa_1 + 2k^2\}} .$$

The correctness of formula (1) was checked by comparing the results with numerical integrations of the differential cross section. Complete agreement was reached in the total energy range.

The contribution to the cross section of the second part of (1) containing the integral is decreasing with increasing energies of the incoming particles. Therefore the first six terms of $\mathrm{d}\sigma/\mathrm{d}k$ represent a good approximation at extremely relativistic energies except near the short-wavelength limit $k = k_{\max} = p^2/\epsilon$, and the accuracy of the numerical integration needs not be very high.

It is useful to derive from (1) approximation formulae for the low-energy end of the photon spectrum, $k \ll k_{\max}$. Expanding the cross section (1) into powers of k, it is possible to evaluate the remaining integrals analytically. Up to relative order $(k/k_{\max})^2$ one gets

$$\begin{aligned}
\frac{k}{\alpha r_0^2}\,\frac{\mathrm{d}\sigma}{\mathrm{d}k}
&\approx \left(\frac{\epsilon^2 + p^2}{\epsilon^2 p^2}\right)^2 \left[\tfrac{8}{3}\epsilon^2 p^2 + \epsilon^2 + p^2 - \frac{\ln(\epsilon + p)}{\epsilon p}\right] \ln\frac{4\epsilon p^2}{k} \\
&- \frac{2}{p^2}\left(16 - 1/\epsilon^4\right)\ln^2(\epsilon + p) \\
&+ \frac{\ln(\epsilon + p)}{3\epsilon p}\left(104 - \frac{2}{\epsilon^2} + \frac{54}{p^2} - \frac{8}{\epsilon^4} - \frac{1}{\epsilon^6}\right) - \frac{16}{3} - \frac{17}{\epsilon^2} \\
&- \frac{18}{p^2} - \frac{6}{\epsilon^4} - \frac{p^4}{3\epsilon^6} + \frac{\epsilon^2 + p^2}{\epsilon p^2}\left(4 - 0.5/\epsilon^4\right) I(\epsilon) \\
&+ \frac{k}{\epsilon p^2}\Bigg\{-\Bigg[\frac{32}{3}\epsilon^2 - \frac{4}{3} - \frac{2}{\epsilon^2 p^2} - \frac{3}{4\epsilon^4 p^4} \\
&\quad + \frac{\epsilon^2 + p^2}{\epsilon^3 p^3}\left(1 + \frac{3}{4\epsilon^2 p^2}\right)\ln(\epsilon + p)\Bigg] \ln\frac{4\epsilon p^2}{k} \\
&+ \left(16 + 1/\epsilon^4\right)\ln^2(\epsilon + p) \\
&+ \frac{\ln(\epsilon + p)}{\epsilon p} \\
&\times \left(\frac{10}{3} - \frac{28}{3}p^2 + \frac{5}{6\epsilon^2} + \frac{2}{3\epsilon^4} + \frac{5}{12\epsilon^6} + \frac{13}{2p^2} + \frac{1}{2p^4}\right) \\
&- \frac{16}{3}p^2 - \frac{5}{3} - \frac{7}{p^2} - \frac{11}{6\epsilon^2 p^2} - \frac{1}{2p^4} + \frac{8}{3\epsilon^4} + \frac{5}{12\epsilon^6} \\
&- \frac{I(\epsilon)}{\epsilon}\left[2\left(\epsilon^2 + p^2\right) - \frac{7}{8p^2} + \frac{3p^2}{8\epsilon^4}\right]\Bigg\} \\
&+ \frac{k^2}{\epsilon^2 p^2}\Bigg\{\ln\frac{4\epsilon p^2}{k} \\
&\times \Bigg[8\epsilon^2 + \frac{20}{3} + \frac{17}{24\epsilon^2 p^2} + \frac{1}{8\epsilon^4} + \frac{40\epsilon^2}{3p^4} - \frac{3}{2p^4} + \frac{5}{8p^6} \\
&\quad - \frac{\ln(\epsilon + p)}{\epsilon p}\left(8\epsilon^2 + 10 + \frac{85}{4p^2} + \frac{p^2}{8\epsilon^4} + \frac{11}{p^4} + \frac{5\epsilon^4}{8p^6}\right)\Bigg] \\
&+ \frac{4}{3}\epsilon^2 - 3 - \frac{29}{2p^2} - \frac{41}{4p^4} - \frac{9}{16p^6} \\
&+ \frac{9}{4\epsilon^4} - \frac{47}{24\epsilon^4 p^2} + \frac{35}{48\epsilon^6} \\
&+ \frac{\ln(\epsilon + p)}{\epsilon p} \\
&\times \Bigg(\frac{40}{3}\epsilon^2 + 8 + \frac{22}{p^2} - \frac{19}{\epsilon^2} \\
&\quad + \frac{11}{24\epsilon^4} + \frac{77}{8p^4} + \frac{35}{48\epsilon^6} + \frac{9}{16p^6}\Bigg) \\
&- \left(12 - 1/\epsilon^4\right)\ln^2(\epsilon + p) \\
&+ \frac{I(\epsilon)}{\epsilon}\left(4\epsilon^2 + \frac{3}{2} - \frac{3}{4\epsilon^2} + \frac{13}{4p^2} + \frac{15}{32\epsilon^4} + \frac{105}{32p^4}\right)\Bigg\} ,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
I(\epsilon) &= \int_{-1}^{+1} \frac{\ln(\epsilon + px)}{\epsilon - px}\,\mathrm{d}x \\
&= \frac{1}{p}\Bigg\{\ln^2(2\epsilon) - \ln^2(\epsilon + p) + 2\ln(2\epsilon)\ln(\epsilon + p) \\
&\quad + 2\,\mathrm{Li}_2\left(\frac{\epsilon - p}{2\epsilon}\right) - \frac{\pi^2}{6}\Bigg\} ,
\end{aligned} \tag{3}$$

and $\mathrm{Li}_2(x)$ is the Euler dilogarithm,

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln|1-t|}{t}\,\mathrm{d}t\,. \tag{4}$$

For $p^2 \ll 1$ the integral (3) can be expanded in terms of p^2 yielding

$$I(\epsilon) \approx \tfrac{4}{3}p^2\left(1 - \tfrac{7}{10}p^2 + \tfrac{149}{280}p^4 - \tfrac{2\,161}{5\,040}p^6 + \tfrac{53\,089}{147\,840}p^8\right)\,. \tag{5}$$

The approximation (2) is very accurate at small and moderate photon energies. For $k < 0.3(\epsilon - 1)$ the relative error never exceeds 0.3%. At low kinetic energies of the incoming particles, $E < 50\,\mathrm{keV}$, the error of formula (2) is even less than 0.1% up to photon energies $h\nu = \frac{1}{2}E$.

The fact that (2) represents an expansion of the cross section (1) in terms of $k/k_{\max} = \epsilon k/p^2$ can be clearly seen in the non-relativistic limit $p^2 \ll 1$ where the higher negative powers of p^2 cancel, resulting in

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}k} \approx \frac{\alpha r_0^2}{p^2 k}\Bigg\{&\tfrac{16}{3}\left(1 + \tfrac{12}{5}p^2 - \tfrac{4}{35}p^4 - \tfrac{32}{105}p^6\right)\ln\frac{4p^2}{k}\\ &-32p^2 + \tfrac{616}{15}p^4 - \tfrac{2\,216}{45}p^6\\ &-\frac{k}{p^2}\Bigg[p^2\left(\tfrac{136}{15} + \tfrac{268}{35}p^2 - \tfrac{127}{21}p^4\right)\ln\frac{4p^2}{k}\\ &\quad+\tfrac{8}{3} - \tfrac{104}{15}p^2 + \tfrac{2\,327}{105}p^4 - \tfrac{1\,616}{63}p^6\Bigg]\\ &+\frac{k^2}{p^4}\Bigg[p^4\left(\tfrac{464}{105} + \tfrac{502}{315}p^2 - \tfrac{2\,104}{3\,465}p^4\right)\ln\frac{4p^2}{k}\\ &\quad-1 + \tfrac{24}{5}p^2 - \tfrac{473}{70}p^4 + \tfrac{17\,917}{630}p^6\Bigg]\Bigg\}\,. \end{aligned}\tag{6}$$

To lowest order in p^2 this formula agrees with the long-wavelength limit of the non-relativistic cross section of Garibyan [10]. Likewise, for extreme relativistic energies $\epsilon \gg 1$ formula (2) reduces to

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}k} \approx \frac{32}{3}\frac{\alpha r_0^2}{k}\Bigg\{&\ln\frac{4\epsilon^3}{k} - \frac{1}{2} - \frac{k}{\epsilon}\left(\ln\frac{4\epsilon^3}{k} + \frac{1}{2}\right)\\ &+\frac{k^2}{\epsilon^2}\left(\frac{3}{4}\ln\frac{4\epsilon^3}{k} + \frac{1}{8}\right)\Bigg\}\,,\end{aligned}\tag{7}$$

which is identical to the cross section of Altarelli and Buccella [1], expanded up to terms of order $(k/\epsilon)^2$. These limiting cases again verify the correctness of (1) and (2).

The total energy loss of the incident particles due to EPB is proportional to

$$\mathcal{E} = \int_0^{k_{\max}} k\,\frac{\mathrm{d}\sigma}{\mathrm{d}k}\,\mathrm{d}k\,. \tag{8}$$

Since the integrand is logarithmically divergent at $k = 0$, it is expedient to split the integral into two parts,

$$\mathcal{E} = \int_0^{k_0} k\,\frac{\mathrm{d}\sigma}{\mathrm{d}k}\,\mathrm{d}k + \int_{k_0}^{k_{\max}} k\,\frac{\mathrm{d}\sigma}{\mathrm{d}k}\,\mathrm{d}k\,, \tag{9}$$

where $\mathrm{d}\sigma/\mathrm{d}k$ is given by (2) in the first term and by (1) in the second term, and k_0 is a sufficiently low photon energy that the cross section (2) is still accurate. The first integral of (9) can be easily performed analytically.

The cross sections presented have been derived in first Born approximation. To allow for Coulomb corrections they have to be multiplied by the factor [9]

$$F_{e^-e^+} = \frac{a'}{a}\,\frac{1-\exp(-2\pi a)}{1-\exp(-2\pi a')}\,, \tag{10}$$

where

$$a = \alpha\,\frac{\epsilon^2 + p^2}{2\epsilon p}\,,\quad a' = \alpha\,\frac{\epsilon^2 + p^2 - 2\epsilon k}{2\sqrt{(\epsilon^2 - \epsilon k)\,(p^2 - \epsilon k)}}\,. \tag{11}$$

Due to the small factor $\alpha \approx 1/137$ in the quantities a and a' the effect of the factor $F_{e^-e^+}$ is significant only at low kinetic energies of the electrons and positrons, and especially near the short-wavelength limit $k = k_{\max} = p^2/\epsilon$, where $a' \to \infty$. Here, the uncorrected cross section vanishes. Since some of the coefficients a_i tend to infinity for $k \to k_{\max}$, it is expedient not to use the expression (1) to perform the limit $k \to p^2/\epsilon$. Rather one traces back to the doubly differential cross section [9]. Multiplying $\mathrm{d}^2\sigma/\mathrm{d}k\,\mathrm{d}\Omega$ by the factor $F_{e^-e^+}$, setting $k = p^2/\epsilon$, and integrating it over the photon angles results in the finite cross section at the short-wavelength limit,

$$\begin{aligned}\frac{\mathrm{d}\sigma_0}{\mathrm{d}k} = \alpha r_0^2\,&\frac{1-\mathrm{e}^{-2\pi a}}{p^2\,(\epsilon^2 + p^2)}\\ \times\Bigg\{&\frac{\ln(\epsilon + p)}{p}\left[\epsilon^4 + 17\epsilon^2 - \frac{19}{2} + \frac{1}{\epsilon^2} + \frac{15}{p^2} + \frac{4}{p^4}\right]\\ &+\frac{11}{2}\epsilon^3 - \frac{119}{6}\,\epsilon - \frac{4}{\epsilon} + \frac{1}{6\epsilon^5} - \frac{5\epsilon}{3p^2} - \frac{4\epsilon}{p^4}\Bigg\}\,.\end{aligned}\tag{12}$$

For high energies, $\epsilon \gg 1$, the leading terms of (12),

$$\frac{\mathrm{d}\sigma_0}{\mathrm{d}k} \approx \frac{\alpha r_0^2}{2\epsilon}\left(1 - \mathrm{e}^{-2\pi a}\right)\{\ln(2\epsilon) + 11/2\} \tag{13}$$

agree with the result of Baĭer et al. [4].

3 Results

The cross section (1) is valid for all energies from the non-relativistic regime to extremely relativistic energies. It is most useful at low and moderately relativistic energies since there exists no other formula of comparable accuracy. These energies occur in hot astrophysical plasmas. At highly relativistic particle energies, $E > 50\,\mathrm{MeV}$, the approximation derived by Baĭer et al. [4] is found to be very accurate even near the short-wavelength limit of the spectrum, the deviations being less than 0.07%.

At high incident energies the bremsstrahlung spectrum is rather flat up to the close vicinity of the maximum photon energy where the cross section decreases steeply to the small value given by (12). An example is shown in Fig. 1

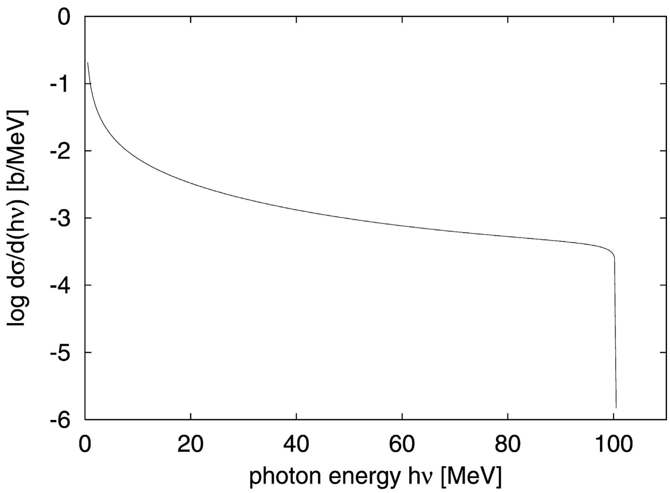

Fig. 1. Spectrum of electron–positron bremsstrahlung for incident particles of energy $E = 100\,\mathrm{MeV}$

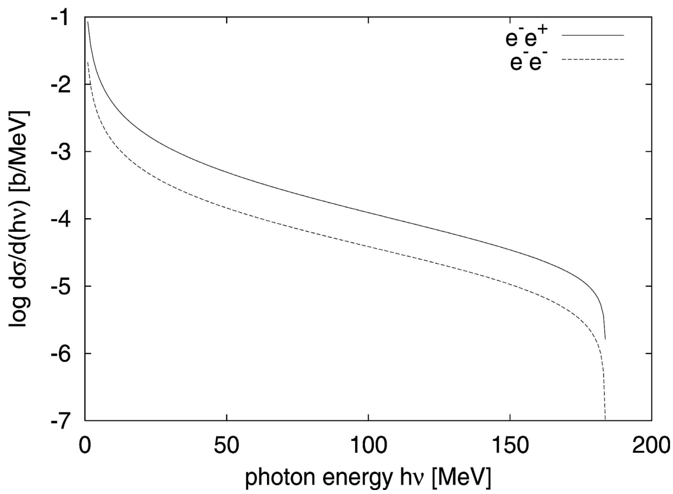

Fig. 2. Spectra of electron–positron (e^-e^+) and electron–electron (e^-e^-) bremsstrahlung for incident particles of energy $E = 100\,\mathrm{keV}$

for $E = 100\,\mathrm{MeV}$. It is interesting to compare the EPB cross section with that of electron–electron (e^-e^-) bremsstrahlung [11]. The two processes differ in their Feynman diagrams. Whereas the scattering graphs are the same, the annihilation diagrams of EPB are replaced by the exchange diagrams of e^-e^- bremsstrahlung. In addition, the electric dipole moment of the e^-e^- system is zero so that its radiation is of quadrupole nature. At ultrarelativistic energies there is no perceptible difference between the spectra except for the very tip [4]. For low energies, on the other hand, the cross section of electron–positron bremsstrahlung exceeds considerably that of electron–electron bremsstrahlung, as is displayed in Fig. 2 for incident energies of $100\,\mathrm{keV}$.

References

1. G. Altarelli, F. Buccella, Nuovo Cimento **34**, 1337 (1964)
2. S.M. Swanson, Phys. Rev. **154**, 1601 (1975)
3. V.N. Baĭer, V.S. Fadin, V.A. Khoze, Sov. Phys. JETP **24**, 760 (1967) [Zh. Eksp. Teor. Fiz. **51**, 1135 (1966)]
4. V.N. Baĭer, V.S. Fadin, V.A. Khoze, Sov. Phys. JETP **26**, 1238 (1968) [Zh. Eksp. Teor. Fiz. **53**, 2194 (1967)]
5. V.N. Baĭer, V.S. Fadin, V.A. Khoze, E.A. Kuraev, Phys. Rep. **78**, 293 (1981)
6. F.A. Berends, R. Kleiss, P. de Causmaecker, R. Gastmans, Tai Tsun Wu, Nucl. Phys. B **206**, 61 (1982)
7. F.A. Berends, R. Kleiss, Nucl. Phys. B **228**, 537 (1983)
8. R. Svensson, Astrophys. J. **258**, 335 (1982)
9. E. Haug, Phys. Rev. D **31**, 2120 (1985); Phys. Rev. D **32**, 1594(E) (1985)
10. G.M. Garibyan, Zh. Eksp. Teor. Fiz. **24**, 617 (1953)
11. E. Haug, Astron. Astrophys. **218**, 330 (1989)